Midterm Exam - Optimization)
B. Math III

19 February, 2025

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 110).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. (15 points) Find the Perron root and the Perron vector for

$$\mathbf{A} = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix}$$

where $\alpha + \beta = 1$ with $\alpha, \beta > 0$. Total for Question 1: 15

2. (15 points) The support function, $S_K : S^{n-1} \to (-\infty, \infty]$, of a closed convex set $K \subseteq \mathbb{R}^n$ is defined as

$$S_K(\vec{y}) = \sup\{\langle \vec{y}, \vec{x} \rangle : \vec{x} \in K\} \leq \infty,$$

for every unit vector $\vec{y}$ in $\mathbb{R}^n$. Suppose that K and L are closed convex sets in $\mathbb{R}^n$. Show that $K = L$ if and only if $S_K = S_L$. Total for Question 2: 15

3. A real (n, n)-matrix $A = ((\alpha_{ij}))$ is called *doubly stochastic* if $\alpha_{ij} \geq 0$ and $\sum_{k=1}^{n} \alpha_{kj} = \sum_{k=1}^{n} \alpha_{ik} = 1$ for $i, j \in \{1, \ldots, n\}$. A doubly stochastic matrix with components in $\{0, 1\}$ is called a permutation matrix.

(a) (5 points) Prove that the set $K \subset \mathbb{R}^{n^2}$ of doubly stochastic matrices is compact and convex.

(b) (10 points) Find, with justification, all extreme points of K.

Total for Question 3: 15

4. (15 points) Consider a set $\mathcal{P}$ described by linear inequality constraints, that is,

$$\mathcal{P} := \{\vec{x} \in \mathbb{R}^n \mid \vec{a}_i^{\,T}\vec{x} \leq b_i, i = 1, \ldots, m\}.$$

A ball with center $\vec{y}$ and radius r is defined as the set of all points within (Euclidean) distance r from $\vec{y}$. We are interested in finding a ball with the largest possible radius, which is entirely contained within the set $\mathcal{P}$. Provide, with justification, a linear programming formulation of this problem.

Total for Question 4: 15

5. Let $\mathbf{c}$ be a vector in $\mathbb{R}^n$. Consider the problem of minimizing $\mathbf{c}^T\mathbf{x}$ where $\mathbf{x}$ varies over a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$.

(a) (10 points) Prove that $\mathbf{x} \in \mathcal{P}$ is optimal if and only if $\mathbf{c}^T\mathbf{d}$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.

(b) (10 points) Prove that $\mathbf{x} \in \mathcal{P}$ is the unique optimal solution if and only if $\mathbf{c}^T\mathbf{d} > 0$ for every non-zero feasible direction $\mathbf{d}$ at $\mathbf{x}$.

Total for Question 5: 20

6. (20 points) Consider a standard form LP problem, under the usual assumption that the rows of $\mathbf{A}$ are linearly independent. Let ϵ be a scalar and define

$$\mathbf{b}(\epsilon) = \mathbf{b} + \begin{bmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^m \end{bmatrix}$$

For every $\epsilon > 0$, we define the ϵ-perturbed problem to be the linear programming problem obtained by replacing b with $b(\epsilon)$. Show that there exists some $\epsilon^* > 0$ such that all basic solutions to the ϵ-perturbed problem are nondegenerate, for $0 < \epsilon < \epsilon^*$.

Total for Question 6: 20

7. (10 points) Consider the problem

$$\begin{array}{ll} \text{minimize} & -2x_1 - x_2 \\ \text{subject to} & x_1 - x_2 \leq 2 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0. \end{array}$$

Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.

Total for Question 7: 10